

ON SPLITTING OF EXACT DIFFERENTIAL FORMS

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Abstract. In work the internal structure of de Rham cohomology is considered. As examples the phase flows in \mathbb{R}^3 admitting the Nambu poisson structure are studied.

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1 Introduction

Let $\Lambda^k(\mathbb{M})$ - be the exterior graded algebra of differential forms with de Rham complex

$$0 \rightarrow \Lambda^0(\mathbb{M}) \rightarrow \Lambda^1(\mathbb{M}) \rightarrow \dots \rightarrow \Lambda^{n-1}(\mathbb{M}) \rightarrow \Lambda^n(\mathbb{M}) \rightarrow 0.$$

Remember [1], that differential form $\omega \in \Lambda^k(\mathbb{M})$ is called closed if $d\omega = 0$, and exact if $\omega = d\nu$ for some $\nu \in \Lambda^{k-1}(\mathbb{M})$. The quotient of closed k -forms on manifolds \mathbb{M} by exact k -forms is the k 'th de Rham cohomology group

$$H^k(\mathbb{M}) = \frac{\ker(d : \Lambda^k(\mathbb{M}) \rightarrow \Lambda^{k+1}(\mathbb{M}))}{\text{im}(d : \Lambda^{k-1}(\mathbb{M}) \rightarrow \Lambda^k(\mathbb{M}))}.$$

It is known, that cohomology $\mathbb{M} = \mathbb{R}^n$ is vanish. It means, that for anyone $\omega \in \Lambda^k(\mathbb{R}^n)$ such that $d\omega = 0$ there exists a $\nu \in \Lambda^{k-1}(\mathbb{R}^n)$ such that $\omega = d\nu$. In any case the dimension of cohomology group is determined by Betti numbers:

$$b^k = \frac{\dim(\omega : d\omega = 0)}{\dim(d\nu)}.$$

2 Cogomology in $\Lambda^2(\mathbb{R}^n)$

Let's connect with de Ram complex the differential module $\{C, d\}$, then

$$Z(C) = \ker d = \{x \in C \mid dx = 0\}$$

are called cocycles of module $\{C, d\}$ (space of closed forms),

$$B(C) = \text{im}d = dC = \{x = dy \mid y \in C\}$$

are called coboundary of module $\{C, d\}$ (space of exact forms). In given designations the group i -cohomology H^i be the quotient i -cocycles by i -coboundary

$$H^i = Z^i / B^i.$$

Assume that $\omega \in B^2 \subset \Lambda^2(\mathbb{R}^n)$. This means that

$$d\omega = 0, \quad \omega = d\nu, \quad \text{where } \nu \in \Lambda^1(\mathbb{R}^n).$$

The received quotient we can write as

$$H_2^2(\mathbb{R}^n) = \frac{\ker(d : \Lambda^2(\mathbb{R}^n) \rightarrow \Lambda^3(\mathbb{R}^n))}{\text{im}(d : \Lambda^1(\mathbb{R}^n) \rightarrow \Lambda^2(\mathbb{R}^n))} = Z^2 / B^2.$$

Note however that in $\Lambda^2(\mathbb{R}^n)$ there exists a form

$$\omega = \lambda_1 \wedge \lambda_2, \quad \text{where } \lambda_1, \lambda_2 \in \Lambda^1(\mathbb{R}^n) \quad \text{such that } d\lambda_i = 0.$$

Let $\lambda_i \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda_i = d\mu_i$), then

$$d\omega = 0, \quad \omega = d\mu_1 \wedge d\mu_2.$$

In other words, exact $\omega \in \Lambda^2(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this space as

$$B^{1,1} = \{x = dy_1 \wedge dy_2 \mid y_i \in C\}$$

then

$$H_{1,1}^2(\mathbb{R}^n) = \frac{\ker(d : \Lambda^2(\mathbb{R}^n) \rightarrow \Lambda^3(\mathbb{R}^n))}{\text{im}(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n)) \oplus \text{im}(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n))} = Z^2 / B^{1,1},$$

or

$$b_{1,1}^2(\mathbb{R}^n) = \frac{\dim(\omega : d\omega = 0)}{\dim(d\mu \wedge d\mu)}.$$

It is obvious that from $B^2 = \{d\mu_1 \wedge d\mu_2, d\nu\}$, and $B^{1,1} = \{d\mu_1 \wedge d\mu_2\}$ it follows that $B^{1,1} \subset B^2$. This means that quotient

$$B^2 / B^{1,1} \simeq H_{1,1}^2 / H_2^2$$

should characterize presence of obstacles (topological defects) for existence of the exact forms in $\Lambda^2(\mathbb{R}^n)$, which are wedge product of exact form from $\Lambda^1(\mathbb{R}^n)$.

Example 1. Consider the dynamical systems in \mathbb{R}^3

$$\dot{x} = -xz; \quad \dot{y} = yz; \quad \dot{z} = x^2 - y^2.$$

According [2], this phase flow has one vectorial

$$\mathbf{h} = \frac{1}{4} ((-x^2y + y^3 + yz^2)dx + (x^3 - y^2x + xz^2)dy - 2xyzdz)$$

and two scalar Hamiltonians

$$H = \frac{1}{2}(x^2 + y^2 + z^2), \quad F = xy.$$

These Hamiltonians are connected by expressions

$$d\mathbf{h} = dH \wedge dF.$$

This means that our system admit Poisson structure with vectorial Hamiltonian

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i,$$

where

$$X_h = -xz \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} + (x^2 - y^2) \frac{\partial}{\partial z},$$

and Poisson structure (Nambu [3])

$$\dot{x}_i = \{H, F, x_i\} = X_H \rfloor dF \wedge dx_i = -X_F \rfloor dH \wedge dx_i,$$

where

$$\begin{aligned} X_H &= z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}, \\ X_F &= y \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + x \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}. \end{aligned}$$

Example 2. Divergence-free Lorenz set

$$\dot{x} = y - z; \quad \dot{y} = -x + xz; \quad \dot{z} = x - xy$$

has one vectorial

$$\begin{aligned}\mathbf{h} = & \left(\frac{x}{4}(z^2 + y^2) - \frac{x}{3}(z + y) \right) dx \\ & + \left(\frac{1}{3}(x^2 - yz + z^2) - \frac{1}{4}x^2y \right) dy \\ & + \left(\frac{1}{3}(y^2 - zy + x^2) - \frac{1}{4}x^2z \right) dz\end{aligned}$$

and two scalar Hamiltonians

$$H = \frac{1}{2}(x^2 + y^2 + z^2), \quad F = \left(y - \frac{y^2}{2} \right) + \left(z - \frac{z^2}{2} \right)$$

connected by expressions

$$d\mathbf{h} = dH \wedge dF.$$

This means that our system admit Poisson structure with vectorial Hamiltonian

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i$$

where

$$X_h = (y - z) \frac{\partial}{\partial x} + (-x + xz) \frac{\partial}{\partial y} + (x - xy) \frac{\partial}{\partial z},$$

and Poisson structure in two forms

$$\dot{x}_i = \{H, F, x_i\} = X_H \rfloor dF \wedge dx_i = -X_F \rfloor dH \wedge dx_i,$$

where

$$\begin{aligned}X_H &= z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}, \\ X_F &= (1 - z) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + (1 - y) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.\end{aligned}$$

Example 3. Phase flow

$$\dot{x} = xy; \quad \dot{y} = x - z; \quad \dot{z} = -zy$$

has one vectorial

$$\mathbf{h} = \frac{1}{12} \begin{pmatrix} z(3y^2 + 4x - 4z) \\ -6xyz \\ x(3y + 4z - 4x) \end{pmatrix},$$

one scalar Hamiltonian

$$H = x - \frac{y^2}{2} + z$$

and prehamiltonian form

$$\Theta = -zdx + xdz$$

connected by expressions

$$d\mathbf{h} = dH \wedge \Theta.$$

This means that our system admit Poisson structure with vectorial Hamiltonian

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i$$

where

$$X_h = xy \frac{\partial}{\partial x} + (x - z) \frac{\partial}{\partial y} - yz \frac{\partial}{\partial z},$$

and scalar Poisson structure in the form

$$\dot{x}_i = X_H \rfloor \Theta \wedge dx_i,$$

where

$$X_H = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$

Therefore,

$$dh \in B^2 \setminus B^{1,1}.$$

For completeness of a statement we shall notice that as $d\Theta \neq 0$, but $d\Theta \wedge \Theta = 0$, then Pfaff equation on prehamiltonians form Θ has solved with integrating factor

$$dF = \frac{\Theta}{x^2 + z^2}, \quad \Rightarrow \quad F = \arctan \frac{z}{x}.$$

The caused of global non-integrability of the given system is the presence holes ($x = 0, z = 0$) in $x0z$ planes. It is obvious that vanishing of second Hamiltonians has not admitted to enter of Nambu structure with a bracket $\{H, F, G\}$.

3 Cogomology in $\Lambda^3(\mathbb{R}^n)$

Further, the top index of the any form will denote its degree, i.e. $\omega^k \in \Lambda^k(\mathbb{R}^n)$. For standard de Rham complexes in \mathbb{R}^n we get $\omega^3 \in B^3 \subset \Lambda^3(\mathbb{R}^n)$. This means that

$$d\omega^3 = 0, \quad \omega^3 = d\nu^2, \quad \text{where } \nu^2 \in \Lambda^2(\mathbb{R}^n).$$

Thus

$$\begin{aligned} H_3^3(\mathbb{R}^n) &= \frac{\ker(d : \Lambda^3(\mathbb{R}^n) \rightarrow \Lambda^4(\mathbb{R}^n))}{\text{im}(d : \Lambda^2(\mathbb{R}^n) \rightarrow \Lambda^3(\mathbb{R}^n))} = Z^3/B^3, \\ b_3^3 &= \frac{\dim(\omega^3 : d\omega^3 = 0)}{\dim(d\nu^2)}. \end{aligned}$$

But in $\Lambda^3(\mathbb{R}^n)$ there exists are forms

$$\omega_1^3 = \lambda_1^1 \wedge \lambda_2^1 \wedge \lambda_3^1, \quad \text{where } \lambda_i^1 \in \Lambda^1(\mathbb{R}^n), \quad \text{such that } d\lambda_i^1 = 0.$$

Let $\lambda_i^1 \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda_i^1 = d\mu_i^0$), then

$$d\omega_1^3 = 0, \quad \omega_1^3 = d\mu_1^0 \wedge d\mu_2^0 \wedge d\mu_3^0.$$

In other words, exact $\omega_1^3 \in \Lambda^3(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this quotient as

$$\begin{aligned} H_{1,1,1}^3(\mathbb{R}^n) &= \frac{\ker(d : \Lambda^3(\mathbb{R}^n) \rightarrow \Lambda^4(\mathbb{R}^n))}{\bigoplus_{k=1}^3 \text{im}_k(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n))} = Z^3/B^{1,1,1}, \\ b_{1,1,1}^3 &= \frac{\dim(\omega^3 : d\omega^3 = 0)}{\dim(d\mu^0 \wedge d\mu^0 \wedge d\mu^0)}. \end{aligned}$$

At the same time in $\Lambda^3(\mathbb{R}^n)$ it is possible also to construct the forms

$$\omega_2^3 = \lambda^1 \wedge \lambda^2, \quad \text{where } \lambda^k \in \Lambda^k(\mathbb{R}^n)$$

such that $d\lambda^k = 0$. Let $\lambda^k \notin H^k(\mathbb{R}^n)$ (i.e. $\lambda_i^k = d\mu_i^k$), then

$$d\omega_2^3 = 0, \quad \omega_2^3 = d\mu^0 \wedge d\mu^1.$$

In other words, exact ω_2^3 be wedge product of the other exact forms. We can write this quotient as

$$H_{2,1}^3(\mathbb{R}^n) = \frac{\ker(d : \Lambda^3(\mathbb{R}^n) \rightarrow \Lambda^4(\mathbb{R}^n))}{\text{im}(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n)) \oplus \text{im}(d : \Lambda^1(\mathbb{R}^n) \rightarrow \Lambda^2(\mathbb{R}^n))} = Z^3/B^{2,1},$$

$$b_{1,2}^3 = \frac{\dim(\omega^3 : d\omega^3 = 0)}{\dim(d\mu^0 \wedge d\mu^1)}.$$

Evidently that

$$H_{1,1,1}^3(\mathbb{R}^n) \supset H_{1,2}^3(\mathbb{R}^n) \supset H_3^3(\mathbb{R}^n),$$

such that quotients $H_{1,1,1}^3(\mathbb{R}^n)/H_3^3(\mathbb{R}^n)$ and $H_{1,1,1}^3(\mathbb{R}^n)/H_{2,1}^3(\mathbb{R}^n)$ should characterize presence of obstacles (topological defects) for existence of the exact forms in $\Lambda^3(\mathbb{R}^n)$, which are wedge product of exact form from $\Lambda^1(\mathbb{R}^n)$ or from $\Lambda^2(\mathbb{R}^n)$.

4 Cogomology in $\Lambda^k(\mathbb{R}^n)$

Generalizing the previous calculations we consider de Rham complex in \mathbb{R}^n and get $\omega^k \in \Lambda^k(\mathbb{R}^n) \notin H^k(\mathbb{R}^n)$. Then

$$d\omega^k = 0, \quad \omega^k = d\nu^{k-1}, \quad \text{where } \nu^{k-1} \in \Lambda^{k-1}(\mathbb{R}^n),$$

and

$$H_k^k(\mathbb{R}^n) = \frac{\ker(d : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n))}{\text{im}(d : \Lambda^{k-1}(\mathbb{R}^n) \rightarrow \Lambda^k(\mathbb{R}^n))} = Z^k/B^k.$$

However, into $\Lambda^k(\mathbb{R}^n)$ there exists are forms

$$\omega^k = \bigwedge_{i=1}^k \lambda_i^1, \quad \text{where } \lambda^1 \in \Lambda^1(\mathbb{R}^n)$$

such that $d\lambda^1 = 0$. Suppose that $\lambda^1 \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda^1 = d\mu^0$), then

$$d\omega^k = 0, \quad \omega^k = \bigwedge_{i=1}^k d\mu_i^0.$$

In other words, exact $\omega^k \in \Lambda^k(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this quotient as

$$H_{1,1,\dots,1}^k(\mathbb{R}^n) = \frac{\ker(d : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n))}{\bigoplus_{i=1}^k \text{im}_i(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n))} = Z^k/B^{1,1,\dots,1},$$

$$b_{1,1,\dots,1}^k = \frac{\dim(\omega^k : d\omega^k = 0)}{\dim(\bigwedge_{i=1}^k d\mu_i)}.$$

Continuing it is similarly we receive:
for $\omega^k = d\mu^1 \wedge \bigwedge_{i=1}^{k-2} d\mu_i^0$

$$H_{2,1,1,\dots,1}^k(\mathbb{R}^n) = \frac{\ker(d : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n))}{\text{im}(d : \Lambda^1(\mathbb{R}^n) \rightarrow \Lambda^2(\mathbb{R}^n)) \bigoplus_{i=1}^{k-2} \text{im}_i(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n))},$$

$$b_{2,1,1,\dots,1}^k = \frac{\dim(\omega^k : d\omega^k = 0)}{\dim(d\mu^1 \wedge \bigwedge_{i=1}^{k-2} d\mu_i^0)},$$

for $\omega^k = d\mu^2 \wedge \bigwedge_{i=1}^{k-3} d\mu_i^0$

$$H_{3,1,1,\dots,1}^k(\mathbb{R}^n) = \frac{\ker(d : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n))}{\text{im}(d : \Lambda^2(\mathbb{R}^n) \rightarrow \Lambda^3(\mathbb{R}^n)) \bigoplus_{i=1}^{k-3} \text{im}_i(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n))},$$

where

$$b_{3,1,1,\dots,1}^k = \frac{\dim(\omega^k : d\omega^k = 0)}{\dim(d\mu^2 \wedge \bigwedge_{i=1}^{k-3} d\mu_i^0)};$$

for $\omega^k = d\mu^3 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0$

$$H_{4,1,1,\dots,1}^k(\mathbb{R}^n) = \frac{\ker(d : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n))}{\text{im}(d : \Lambda^3(\mathbb{R}^n) \rightarrow \Lambda^4(\mathbb{R}^n)) \bigoplus_{i=1}^{k-4} \text{im}_i(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n))},$$

where

$$b_{4,1,1,\dots,1}^k = \frac{\dim(\omega^k : d\omega^k = 0)}{\dim(d\mu^3 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0)};$$

...

for $\omega^k = d\mu^1 \wedge d\mu^1 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0$

$$H_{2,2,1,\dots,1}^k(\mathbb{R}^n) = \frac{\ker(d : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n))}{\bigoplus_{i=1}^2 \text{im}_i(d : \Lambda^1(\mathbb{R}^n) \rightarrow \Lambda^2(\mathbb{R}^n)) \bigoplus_{i=1}^{k-4} \text{im}_i(d : \Lambda^0(\mathbb{R}^n) \rightarrow \Lambda^1(\mathbb{R}^n))},$$

where

$$b_{2,2,1,1,\dots,1}^k = \frac{\dim(\omega^k : d\omega^k = 0)}{\dim(d\mu^1 \wedge d\mu^1 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0)}.$$

...

etc.

For simplification of record we shall enter a multiindex

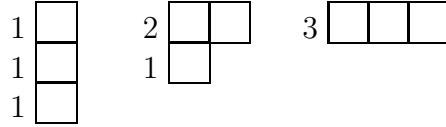
$$\#m = \{m_1, m_2, \dots, m_i\}, \quad m_1 + m_2 + \dots + m_i = k, \quad m_1 \geq m_2 \geq \dots \geq m_i \geq 0,$$

which is formed by a rule of construction of the Young diagrams. Then

$$H_{\#m}^k(\mathbb{R}^n) = \frac{\ker(d : \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{k+1}(\mathbb{R}^n))}{\bigoplus_{\#m} \text{im}_{\#m}(d : \Lambda^{\#m}(\mathbb{R}^n) \rightarrow \Lambda^{\#m+1}(\mathbb{R}^n))} = Z^k / B^{\#m},$$

$$b_{\#m}^k = \frac{\dim(\omega^k : d\omega^k = 0)}{\dim(\bigwedge^{\#m} d\mu_i^0)}.$$

So, for $k = 3$ we shall receive



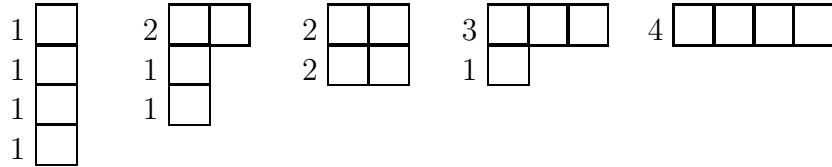
i.e.

$$\begin{aligned} \#m &= \{1, 1, 1\} & \text{or} & H_{\#m}^3(\mathbb{R}^n) = H_{1,1,1}^3(\mathbb{R}^n), \\ \#m &= \{2, 1\} & \text{or} & H_{\#m}^3(\mathbb{R}^n) = H_{2,1}^3(\mathbb{R}^n), \\ \#m &= \{3\} & \text{or} & H_{\#m}^3(\mathbb{R}^n) = H_3^3(\mathbb{R}^n). \end{aligned}$$

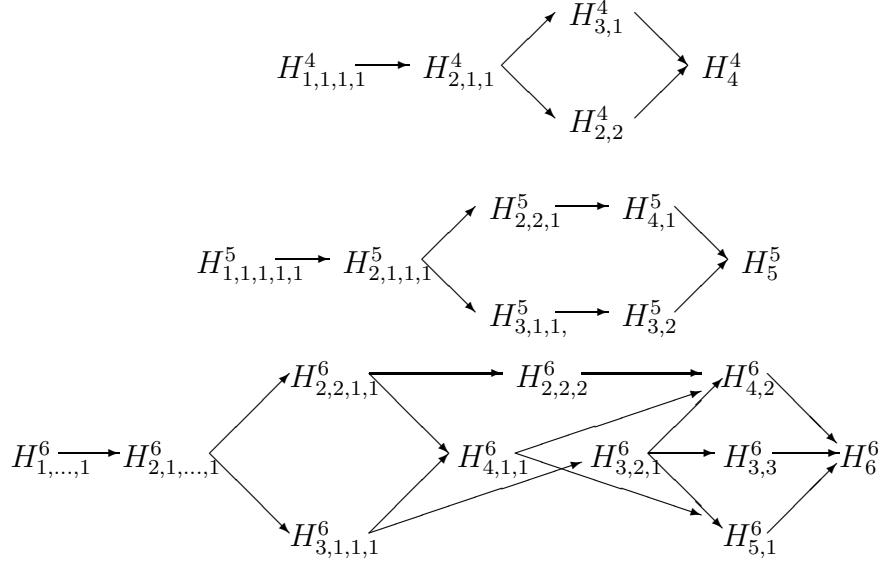
Then the filtered complex of cohomology can be represented as follows

$$H_{1,1,1}^3 \rightarrow H_{2,1}^3 \rightarrow H_3^3.$$

For $k = 4$ we get



i.e. $\#m = \{1, 1, 1, 1\}$, $\#m = \{2, 1, 1\}$, $\#m = \{2, 2\}$, $\#m = \{3, 1\}$ or $\#m = \{4\}$. Let's notice, that at $k \geq 4$ the structure of cohomology $H_{\#m}^k(\mathbb{R}^n)$ is not linear and for some small $k = 4, 5, 6$ is shown in figures:



We can see that in the general case the cohomology sequences are not filtered. This means to define cohomology of cogomology is obviously impossible.

References

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